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## Quantum corrections to the semiclassical quantization of a nonintegrable system

Luca Salasnich<sup>(\*)<sup>(+)</sup>1</sup> and Marko Robnik<sup>(\*)<sup>(+)</sup>2</sup>

<sup>(\*)</sup> Center for Applied Mathematics and Theoretical Physics,  
University of Maribor, Krekova 2, SLO-62000 Maribor, Slovenia

<sup>(+)</sup> Dipartimento di Fisica "G. Galilei" della Università di Padova,  
Istituto Nazionale di Fisica Nucleare, Sezione di Padova,  
Via Marzolo 8, I-35131 Padova, Italy

**Abstract.** We study the semiclassical behaviour of a two-dimensional nonintegrable system. In particular we analyze the question of quantum corrections to the semiclassical quantization obtaining up to the second order of perturbation theory an explicit analytical formula for the energy levels, which is the usual semiclassical one plus quantum corrections. We compare the "exact" levels obtained numerically to the semiclassical levels studying also the effects of quantum corrections.

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<sup>1</sup>e-mail: luca.salasnich@uni-mb.si

<sup>2</sup>e-mail: robnik@uni-mb.si

Nowadays there is considerable renewed interest in the transition from classical mechanics to quantum mechanics, a powerful motivation behind that being the problem of the so-called quantum chaos (see Ozorio de Almeida 1990, Gutzwiller 1990, Casati and Chirikov 1995). An important aspect is represented by the semiclassical quantization formula of the (regular) energy levels for quasi-integrable systems (Maslov and Fredoriuk 1981, Rau 1992, Braun 1993), the so-called torus quantization, initiated by Einstein (1917) and completed by Maslov (1972).

It has been recently shown (Graffi and Paul 1987, Degli Esposti, Graffi and Herczynski 1991) that, for perturbed non-resonant harmonic oscillators, the algorithm of classical perturbation theory can be used to formulate the quantum mechanical perturbation theory as the semiclassically quantized classical perturbation theory equipped with the quantum corrections in powers of  $\hbar$  "correcting" the classical Hamiltonian that appears in the classical algorithm. In effect, one can explicitly calculate corrections to the Einstein–Brillouin–Keller (EBK) quantization of the classical tori (Maslov and Fredouriuk 1981). For example, the quantum corrections of the one-dimensional  $x^4$  perturbed harmonic oscillator have been studied in great details by Alvarez, Graffi and Silverstone (1988).

Examples of rather detailed studies of semiclassical approximations, their resummations and of the Birkhoff–Gustavson normal forms can be found in Ali, Wood and Devitt (1986) and Ali and Devitt (1989). Another example of a rather complete semiclassical analysis of a one-dimensional system, namely the quartic oscillator, has been published by Voros (1983).

The aim of this paper is to extend previous studies to a two-dimensional system, which is more interesting because it is nonintegrable and thus generic. The integrable systems are rather exceptional in the sense that they are typically isolated points in the functional space of Hamiltonians and their measure is zero in this space. If we randomly choose a system realized in nature, the probability is one that the system is nonintegrable (see Robnik 1995).

The model is given by two non-resonant oscillators coupled by a nonlinear quartic interaction of strength  $g$  (Pullen and Edmonds 1981):

$$H = \frac{\omega_1}{2}(p_1^2 + q_1^2) + \frac{\omega_2}{2}(p_2^2 + q_2^2) + gq_1^2q_2^2. \quad (1)$$

Note that a similar Hamiltonian has been obtained for the Yang–Mills–Higgs

quantum mechanics (Savvidy 1984, Salasnich 1995).

Through the canonical transformation in action–angle variables (see Born 1927 and Dittrich and Reuter 1992):

$$q_k = \sqrt{2I_k} \cos \theta_k, \quad p_k = \sqrt{2I_k} \sin \theta_k, \quad k = 1, 2 \quad (2)$$

the Hamiltonian can be written:

$$H = H_0(I_1, I_2) + gV(I_1, I_2, \theta_1, \theta_2), \quad (3)$$

where:

$$H_0(I_1, I_2) = \omega_1 I_1 + \omega_2 I_2, \quad (4)$$

$$V(I_1, I_2, \theta_1, \theta_2) = 4I_1 I_2 \cos^2 \theta_1 \cos^2 \theta_2. \quad (5)$$

Following the classical perturbation theory (see Born 1927, Dittrich and Reuter 1992), we search for a canonical transformation  $(I_1, I_2, \theta_1, \theta_2) \rightarrow (\tilde{I}_1, \tilde{I}_2, \tilde{\theta}_1, \tilde{\theta}_2)$  to obtain a new Hamiltonian that depends only on the new action variables up to the second order in a power series of  $g$ :

$$\tilde{H}(\tilde{I}_1, \tilde{I}_2) = \tilde{H}_0(\tilde{I}_1, \tilde{I}_2) + g\tilde{H}_1(\tilde{I}_1, \tilde{I}_2) + g^2\tilde{H}_2(\tilde{I}_1, \tilde{I}_2). \quad (6)$$

The generator  $S$  of the canonical transformation is supposed to be capable of expansion as a power series in  $g$  of the form:

$$S(\tilde{I}_1, \tilde{I}_2, \theta_1, \theta_2) = \tilde{I}_1 \theta_1 + \tilde{I}_2 \theta_2 + gS_1(\tilde{I}_1, \tilde{I}_2, \theta_1, \theta_2) + g^2S_2(\tilde{I}_1, \tilde{I}_2, \theta_1, \theta_2), \quad (7)$$

and to satisfy the equations:

$$I_k = \frac{\partial S}{\partial \theta_k} = \tilde{I}_k + g \frac{\partial S_1}{\partial \theta_k} + g^2 \frac{\partial S_2}{\partial \theta_k}, \quad (8)$$

$$\tilde{\theta}_k = \frac{\partial S}{\partial \tilde{I}_k} = \theta_k + g \frac{\partial S_1}{\partial \tilde{I}_k} + g^2 \frac{\partial S_2}{\partial \tilde{I}_k}. \quad (9)$$

From the Hamilton–Jacobi equation:

$$H_0\left(\frac{\partial S}{\partial \theta_1}, \frac{\partial S}{\partial \theta_2}\right) + V\left(\frac{\partial S}{\partial \theta_1}, \frac{\partial S}{\partial \theta_2}, \theta_1, \theta_2\right) = \tilde{H}_0(\tilde{I}_1, \tilde{I}_2) + g\tilde{H}_1(\tilde{I}_1, \tilde{I}_2) + g^2\tilde{H}_2(\tilde{I}_1, \tilde{I}_2), \quad (10)$$

we have a number of differential equations that result on equating the coefficients of the powers of  $g$ :

$$\tilde{H}_0(\tilde{I}_1, \tilde{I}_2) = H_0(\tilde{I}_1, \tilde{I}_2) = \omega_1 \tilde{I}_1 + \omega_2 \tilde{I}_2, \quad (11)$$

$$\tilde{H}_1(\tilde{I}_1, \tilde{I}_2) = (\omega_1 \frac{\partial S_1}{\partial \theta_1} + \omega_2 \frac{\partial S_1}{\partial \theta_2}) + V(\tilde{I}_1, \tilde{I}_2, \theta_1, \theta_2), \quad (12)$$

$$\tilde{H}_2(\tilde{I}_1, \tilde{I}_2) = (\omega_1 \frac{\partial S_2}{\partial \theta_1} + \omega_2 \frac{\partial S_2}{\partial \theta_2}) + (\frac{\partial V}{\partial I_1} \frac{\partial S_1}{\partial \theta_1} + \frac{\partial V}{\partial I_2} \frac{\partial S_1}{\partial \theta_2}). \quad (13)$$

The unknown functions  $\tilde{H}_1$ ,  $S_1$ ,  $\tilde{H}_2$  and  $S_2$  may be determined by averaging over the time variation of the unperturbed motion.

At the first order in  $g$  we obtain:

$$\tilde{H}_1(\tilde{I}_1, \tilde{I}_2) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 V(\tilde{I}_1, \tilde{I}_2, \theta_1, \theta_2) = \tilde{I}_1 \tilde{I}_2, \quad (14)$$

and

$$\begin{aligned} S_1(\tilde{I}_1, \tilde{I}_2, \theta_1, \theta_2) = & -\frac{1}{4} \tilde{I}_1 \tilde{I}_2 [\frac{2}{\omega_1} \sin 2\theta_1 + \frac{2}{\omega_2} \sin 2\theta_2 + \\ & + \frac{1}{\omega_1 - \omega_2} \sin 2(\theta_1 - \theta_2) + \frac{1}{\omega_1 + \omega_2} \sin 2(\theta_1 + \theta_2)]. \end{aligned} \quad (15)$$

At the second order in  $g$  we have:

$$\begin{aligned} \tilde{H}_2(\tilde{I}_1, \tilde{I}_2) = & \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 (\frac{\partial V}{\partial I_1} \frac{\partial S_1}{\partial \theta_1} + \frac{\partial V}{\partial I_2} \frac{\partial S_1}{\partial \theta_2}) \\ = & -\frac{1}{8} \tilde{I}_1 \tilde{I}_2 [4(\frac{\tilde{I}_1}{\omega_2} + \frac{\tilde{I}_2}{\omega_1}) - \frac{(\tilde{I}_1 - \tilde{I}_2)}{\omega_1 - \omega_2} + \frac{(\tilde{I}_1 + \tilde{I}_2)}{\omega_1 + \omega_2}]. \end{aligned} \quad (16)$$

The approximate Hamiltonian (6) depends only on the actions<sup>3</sup>, so that by an application of the EBK rule:

$$\tilde{I}_k = (n_k + \frac{1}{2})\hbar, \quad (17)$$

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<sup>3</sup>Note that the integrable approximate Hamiltonian (6) could be obtained alternatively as the Birkhoff–Gustavson normal form, which is a purely algebraic method of calculating the action variables order by order for perturbed harmonic oscillators with polynomial perturbations (Robnik 1984, Kaluža and Robnik 1992, Kaluža 1993, Robnik 1993).

to equations (11), (14) and (16), we obtain a semiclassical analytical formula of the energy levels.

Now we show how to connect this semiclassical formula with the usual quantum perturbation theory. In quantum mechanics the generalized coordinates satisfy the usual commutation rules  $[\hat{q}_k, \hat{p}_l] = i\hbar\delta_{kl}$ , with  $k, l = 1, 2$ . Introducing the creation and destruction operators:

$$\hat{a}_k = \frac{1}{\sqrt{2\hbar}}(\hat{q}_k + i\hat{p}_k), \quad \hat{a}_k^+ = \frac{1}{\sqrt{2\hbar}}(\hat{q}_k - i\hat{p}_k), \quad (18)$$

the quantum Hamiltonian can be written:

$$\hat{H} = \hat{H}_0 + g\hat{V}, \quad (19)$$

where:

$$\hat{H}_0 = \hbar\omega_1(\hat{a}_1^+\hat{a}_1 + \frac{1}{2}) + \hbar\omega_2(\hat{a}_2^+\hat{a}_2 + \frac{1}{2}), \quad (20)$$

$$\hat{V} = \frac{\hbar^2}{4}(\hat{a}_1 + \hat{a}_1^+)^2(\hat{a}_2 + \hat{a}_2^+)^2. \quad (21)$$

At this point we mention the general problem of quantization (Robnik 1984, Robnik 1988, Abraham and Marsden 1978): There is no unique quantization prescription. If one chooses certain quantization procedure, this will *not* commute in general with the classical canonical transformations. However, the quantization does commute with the *linear* classical canonical transformations. This is exactly the approach implemented in our present case (18), and thus our quantization is equivalent to the coordinate space quantization (the latter one yields the *right* quantum mechanics, whose results agree with experiments).

If  $|n_1 n_2\rangle$  is the basis of the occupation numbers of the two harmonic oscillators, the matrix elements are:

$$\langle n'_1 n'_2 | \hat{H}_0 | n_1 n_2 \rangle = \hbar[\omega_1(n_1 + \frac{1}{2}) + \omega_2(n_2 + \frac{1}{2})]\delta_{n'_1 n_1}\delta_{n_2 n_2}, \quad (22)$$

and:

$$\begin{aligned} \langle n'_1 n'_2 | \hat{V} | n_1 n_2 \rangle &= \frac{\hbar^2}{4} [\sqrt{n_1(n_1 - 1)}\delta_{n'_1 n_1 - 2} + \sqrt{(n_1 + 1)(n_1 + 2)}\delta_{n'_1 n_1 + 2} + (2n_1 + 1)\delta_{n'_1 n_1}] \times \\ &\times [\sqrt{n_2(n_2 - 1)}\delta_{n'_2 n_2 - 2} + \sqrt{(n_2 + 1)(n_2 + 2)}\delta_{n'_2 n_2 + 2} + (2n_2 + 1)\delta_{n'_2 n_2}]. \end{aligned} \quad (23)$$

The Rayleigh–Schrödinger perturbation theory (see Messiah 1962) up to the second order gives us:

$$E(n_1\hbar, n_2\hbar) = E_0(n_1\hbar, n_2\hbar) + gE_1(n_1\hbar, n_2\hbar) + g^2E_2(n_1\hbar, n_2\hbar), \quad (24)$$

where:

$$E_0(n_1\hbar, n_2\hbar) = \hbar[\omega_1(n_1 + \frac{1}{2}) + \omega_2(n_2 + \frac{1}{2})], \quad (25)$$

$$E_1(n_1\hbar, n_2\hbar) = < n_1n_2 | \hat{V} | n_1n_2 >, \quad (26)$$

$$E_2(n_1\hbar, n_2\hbar) = \sum_{\substack{n'_1 n'_2 \\ (n'_1, n'_2) \neq (n_1, n_2)}} \frac{| < n'_1 n'_2 | \hat{V} | n_1 n_2 > |^2}{\hbar[\omega_1(n_1 - n'_1) + \omega_2(n_2 - n'_2)]}. \quad (27)$$

We obtain immediately:

$$E_1(n_1\hbar, n_2\hbar) = \hbar^2(n_1 + \frac{1}{2})(n_2 + \frac{1}{2}), \quad (28)$$

and after some calculations:

$$\begin{aligned} E_2(n_1\hbar, n_2\hbar) = & \frac{\hbar^3}{32} \left[ \frac{n_1(n_1 - 1)n_2(n_2 - 1)}{\omega_1 + \omega_2} - \frac{(n_1 + 1)(n_1 + 2)(n_2 + 1)(n_2 + 2)}{\omega_1 + \omega_2} + \right. \\ & + \frac{n_1(n_1 - 1)(n_2 + 1)(n_2 + 2)}{\omega_1 - \omega_2} - \frac{(n_1 + 1)(n_1 + 2)n_2(n_2 - 1)}{\omega_1 - \omega_2} + \\ & + \frac{n_1(n_1 - 1)(2n_2 + 1)^2}{\omega_1} - \frac{(n_1 + 1)(n_1 + 2)(2n_2 + 1)^2}{\omega_1} + \\ & \left. + \frac{(2n_1 + 1)^2 n_2(n_2 - 1)}{\omega_2} - \frac{(2n_1 + 1)^2 (n_2 + 1)(n_2 + 2)}{\omega_2} \right]. \end{aligned} \quad (29)$$

The zero and first order quantum terms coincide with the semi-classical ones:

$$E_0(n_1\hbar, n_2\hbar) = \tilde{H}_0((n_1 + \frac{1}{2})\hbar, (n_2 + \frac{1}{2})\hbar), \quad (30)$$

$$E_1(n_1\hbar, n_2\hbar) = \tilde{H}_1((n_1 + \frac{1}{2})\hbar, (n_2 + \frac{1}{2})\hbar), \quad (31)$$

and the second order quantum term can be written:

$$E_2(n_1\hbar, n_2\hbar) = \tilde{H}_2((n_1 + \frac{1}{2})\hbar, (n_2 + \frac{1}{2})\hbar) + \hbar^2 Q_2((n_1 + \frac{1}{2})\hbar, (n_2 + \frac{1}{2})\hbar), \quad (32)$$

where:

$$Q_2(\tilde{I}_1, \tilde{I}_2) = -\frac{3}{32} \left[ \frac{(\tilde{I}_1 - \tilde{I}_2)}{\omega_1 - \omega_2} + \frac{(\tilde{I}_1 + \tilde{I}_2)}{\omega_1 + \omega_2} \right]. \quad (33)$$

The quantum series rearranges directly into the classical canonical perturbation series plus quantum corrections proportional to successive powers of  $\hbar$  (see Degli Esposti, Graffi and Herczynski 1991).

The term  $\hbar^2 Q_2$  represents the quantum corrections to the EBK quantization up to the second order of perturbation theory. These quantum corrections depend linearly on quantum numbers. It is in contrast with first order results in a previous paper of Robnik (1984) in which the spectra differ only by an additive constant independent of the quantum numbers (actions).

We compute the "exact" levels with a numerical diagonalization of the truncated matrix of the Hamiltonian (19) in the basis of the unperturbed oscillators (see Graffi, Manfredi and Salasnich 1994). The numerical energy levels depend on the dimension of the truncated matrix: We compute the numerical levels in double precision increasing the matrix dimension until the first 100 levels converge within 8 digits (matrix dimension  $1225 \times 1225$ ).

Then we compare the "exact" levels to the levels of the quantum perturbation theory and to the semiclassical results. A very good agreement is observed for the lowest energy levels (see Table 1).

In table 2 we show the error in units of the mean level spacing  $D$  between the "exact" levels and the levels obtained with semiclassical and quantum perturbation theory.

We observe that the algorithm provided by the appropriate semiclassical quantization is comparable to the algorithm provided by ordinary quantum perturbation theory but the quantal corrections do not always increase the accuracy. Thus for some of the calculated levels the semiclassical quantization gives better results than the quantum perturbation theory. In fact, Rayleigh–Schrödinger perturbation theory diverges for any value of  $g$ , but it has been proved to be Borel summable to the exact energy levels only for some special systems like the anharmonic oscillators with  $f$  degrees of freedom with a polynomial perturbing potential which is asymptotically positive definite (Graffi, Grecchi and Simon 1970, Reed and Simon 1978, Caliceti, Graffi and Maioli 1980).

In Figure 1 we plot the error, in units of the mean level spacing  $D$ , between the "exact" levels and the semiclassical levels. The figure shows some systematics but the prediction of individual levels by EBK quantization

worsens as the quantum number increases, contrary to the naive expectation (see also Prosen and Robnik 1993). As is seen, by decreasing  $\hbar$  from 1 to  $10^{-1}$  the quality of the approximation improves considerably but only for sufficiently low levels (see also Table 3).

If  $\hbar$ , no matter how small, is kept fixed, the EBK quantization (torus quantization) of the individual levels is only a first order approximation of an expansion in  $\hbar$ . Therefore, the accuracy of the approximation decreases for higher levels. To get a good agreement it is necessary, as is well known, to implement the classical limit, i.e.  $\hbar \rightarrow 0$  and  $n_1, n_2 \rightarrow \infty$ , while at the same time keeping the actions  $\tilde{I}_1 = (n_1 + 1/2)\hbar$  and  $\tilde{I}_2 = (n_2 + 1/2)\hbar$  constant (Graffi, Manfredi and Salasnich 1994).

In Figure 2 we plot the error, in units of the mean level spacing  $D$ , between semiclassical levels and levels obtained with quantum perturbation theory. Also in this case the error increases for higher levels.

In conclusion, we have examined the transition between the classical and the quantum mechanics for a two-dimensional, nonintegrable and nonresonant system. Up to the second order of perturbation theory we have decomposed the quantum description into the classical description (i.e. the leading semiclassical term) plus quantum corrections which depend linearly on the quantum numbers. The analytical energy levels are in good agreement with the "exact" numerical ones, the semiclassical quantization is comparable to the quantum perturbation theory, and for some levels the semiclassical quantization gives better results than quantum perturbation theory.

Finally we note that the extraction of quantum corrections for resonant systems is a more intricate procedure; some initial results for perturbed resonant oscillators can be found in Graffi (1992).

The present paper supports the following *general conclusions*.

(c1) The quantum corrections, defined as the difference between the semiclassical and the quantum perturbation terms of the same order, as derived by Degli Esposti, Graffi and Herczynski (1991), certainly vanish in the limit  $\hbar \rightarrow 0$ .

(c2) The above conclusion, however, changes drastically when the quantum corrections are measured in natural units of the mean level spacing  $D$ , which scales as  $D \sim \hbar^f$ , where  $f$  is the number of freedoms: The quantum corrections and their low  $\hbar$ -power terms can even diverge or remain constant. This is implicit in Degli Esposti, Graffi and Herczynski (1991). Probably the same conclusion applies when exact levels are compared to the semiclassi-

cal ones, supported by the results in Prosen and Robnik (1993) and by the results of the present paper.

(c3) Since both the classical and quantum perturbation series typically diverge and thus do not necessarily describe the exact levels, not even after a certain resummation (except for some important notable exceptions previously described (Graffi, Grecchi and Simon 1970)), it is important to compare the semiclassical approximation (and the quantal perturbation results) with the exact spectra, which in general is impossible, since we generally do not have explicit solutions of the Schrödinger problem in a closed form. Therefore, we stress the importance of specific case studies like the present one, in order to get a better understanding of the quality of semiclassical approximations.

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$E^{ex}$	$E^{sc}$	$E^{qp}$
1.230722	1.230990	1.230522
2.275974	2.273214	2.274701
2.689415	2.690856	2.687816
3.316524	3.308447	3.311808
3.820434	3.814018	3.812833
4.146646	4.148302	4.142610
4.354307	4.336609	4.341846
4.937708	4.915967	4.916677
5.359848	5.347322	5.345305
5.390110	5.357700	5.364811
5.603778	5.603248	5.594904
6.047742	5.996702	5.999287
6.424398	6.371719	6.380706
6.546966	6.510986	6.509044
6.897049	6.873125	6.866657
7.062932	7.055694	7.044699
7.152476	7.056224	7.060684
7.457506	7.378668	7.389530
7.723943	7.639295	7.639228
8.144146	8.093505	8.088912
8.253060	8.094533	8.100868
8.435119	8.378546	8.382309
8.489923	8.391429	8.391282
8.528618	8.505639	8.491993
8.892640	8.732248	8.734056

**Table 1:** Comparison between "exact" levels and levels obtained by perturbation theories. First 25 levels.  $E^{ex}$  are "exact" levels,  $E^{sc}$  are semiclassical levels, and  $E^{qp}$  are levels obtained with quantum perturbation theory.  $\hbar = 1$ ,  $g = 10^{-1}$ ,  $\omega_1 = 1$  and  $\omega_2 = \sqrt{2}$ .

$ E^{ex} - E^{sc} /D$	$ E^{ex} - E^{qp} /D$
1.0611359·10 <sup>-3</sup>	1.1284242·10 <sup>-3</sup>
1.5578579·10 <sup>-2</sup>	7.1859419·10 <sup>-3</sup>
8.1338054·10 <sup>-3</sup>	9.0260478·10 <sup>-3</sup>
4.5591835·10 <sup>-2</sup>	2.6613854·10 <sup>-2</sup>
3.6215890·10 <sup>-2</sup>	4.2791300·10 <sup>-2</sup>
9.3476856·10 <sup>-3</sup>	2.2781115·10 <sup>-2</sup>
9.9898852·10 <sup>-2</sup>	7.0337765·10 <sup>-2</sup>
0.1227176	0.1187100
7.0703819·10 <sup>-2</sup>	9.2249520·10 <sup>-2</sup>
0.1829406	0.1428019
2.9902905·10 <sup>-2</sup>	5.0089385·10 <sup>-2</sup>
0.2880960	0.2735053
0.2973495	0.2466222
0.2030921	0.2140520
0.1350395	0.1715501
4.0854741·10 <sup>-2</sup>	0.1029161
0.5432989	0.5181223
0.4450069	0.3836938
0.4778005	0.4781800
0.2858459	0.3117708
0.8948155	0.8590558
0.3193286	0.2980870
0.5559518	0.5567808
0.1297049	0.2067311
0.9053394	0.8951331

**Table 2:** The error measured in units of the mean level spacing  $D$  for the first 25 levels.  $D$  is calculated for the lowest 100 levels.  $E^{ex}$  are "exact" levels,  $E^{sc}$  are semiclassical levels, and  $E^{qp}$  are levels obtained with quantum perturbation theory.  $\hbar = 1$ ,  $g = 10^{-1}$ ,  $\omega_1 = 1$  and  $\omega_2 = \sqrt{2}$ .

$\hbar = 1$	$\hbar = 10^{-1}$
1.0611359·10 <sup>-3</sup>	2.4773894·10 <sup>-5</sup>
1.5578579·10 <sup>-2</sup>	1.0003044·10 <sup>-4</sup>
8.1338054·10 <sup>-3</sup>	1.8136360·10 <sup>-4</sup>
4.5591835·10 <sup>-2</sup>	2.5054353·10 <sup>-4</sup>
3.6215890·10 <sup>-2</sup>	5.6091835·10 <sup>-6</sup>
9.3476856·10 <sup>-3</sup>	3.1972348·10 <sup>-4</sup>
9.9898852·10 <sup>-2</sup>	4.3938606·10 <sup>-4</sup>
0.1227176	2.3371598·10 <sup>-4</sup>
7.0703819·10 <sup>-2</sup>	9.3486393·10 <sup>-5</sup>
0.1829406	7.0301769·10 <sup>-4</sup>
2.9902905·10 <sup>-3</sup>	4.4125578·10 <sup>-4</sup>
0.2880960	6.3570746·10 <sup>-4</sup>
0.2973495	2.3932516·10 <sup>-4</sup>
0.2030921	1.0582660·10 <sup>-3</sup>
0.1350395	1.1966258·10 <sup>-4</sup>
4.0854741·10 <sup>-2</sup>	1.2452388·10 <sup>-3</sup>
0.5432989	5.2726327·10 <sup>-4</sup>
0.4450069	8.1146188·10 <sup>-4</sup>
0.4778005	1.5294374·10 <sup>-3</sup>
0.2858459	3.0663537·10 <sup>-4</sup>
0.8948155	2.1427081·10 <sup>-3</sup>
0.3193286	5.2352381·10 <sup>-5</sup>
0.5559518	1.7238891·10 <sup>-4</sup>
0.1297049	2.1165321·10 <sup>-3</sup>
0.9053394	5.6091836·10 <sup>-4</sup>

**Table 3:** The error measured in units of the mean level spacing  $D$  between "exact" levels and semiclassical levels. First 25 levels.  $D$  is calculated for the lowest 100 levels.  $g = 10^{-1}$ ,  $\omega_1 = 1$  and  $\omega_2 = \sqrt{2}$ .

## Figure Captions

**Figure 1:** The error measured in units of the mean level spacing between "exact" levels and semiclassical levels. First 100 levels.  $g = 10^{-1}$ ,  $\omega_1 = 1$  and  $\omega_2 = \sqrt{2}$ . (a)  $\hbar = 1$ , (b)  $\hbar = 10^{-1}$ .

**Figure 2:** The error measured in units of the mean level spacing between semiclassical levels and levels obtained with quantum perturbation theory. First 100 levels.  $g = 10^{-1}$ ,  $\omega_1 = 1$  and  $\omega_2 = \sqrt{2}$ . (a)  $\hbar = 1$ , (b)  $\hbar = 10^{-1}$ .



